Analytic Functions*

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Zasady: Trzeba wybrac 5 zadan i zaznaczyc, ze maja one liczyc sie do wyniku z kolokwium. Pozostale zadania tez mozna rozwiazac. Kazde takie zadanie policze jak 3 zadania na pracy domowej.

Zadanie 1. Does there exist an analytic function in $\mathbb{D}$ such that $|f(\frac{1}{n})| = e^{-n}$ for all $n = 2, 3, \ldots$?

Zadanie 2. Suppose $|a_k| < 1$ for $k = 1, 2, \ldots, n$. Let $|b| < 1$ and define $F(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}$. Show that the equation $F(z) = b$ has exactly n roots in $\mathbb{D}$, counting multiplicities.

Zadanie 3. Compute $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 \mathrm{d}x$.

Zadanie 4. Suppose that all the roots of $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ have positive imaginary part. Show that the polynomials

$$U(z) = \mathrm{Re}(a_0) + \mathrm{Re}(a_1) z + \ldots + \mathrm{Re}(a_n) z^n, \qquad V(z) = \mathrm{Im}(a_0) + \mathrm{Im}(a_1) z + \ldots + \mathrm{Im}(a_n) z^n$$

are real-rooted.

Zadanie 5. Suppose f is analytic in $\mathbb{D}$ and continuous in $\bar{\mathbb{D}}$. Suppose $|f(z)| > 1$ on $\partial\mathbb{D}$ and $|f(0)| < 1$. Show that f has at least one zero in $\mathbb{D}$.

Zadanie 6. Let $f_n : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be given by $f_n(z) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots + \frac{1}{n! z^n}$. Show that for any $\varepsilon > 0$ there exists n_0 such that for $n \geq n_0$ all the solutions to the equation $f_n(z) = 0$ lie in $\{0 < |z| < \varepsilon\}$.

Zadanie 7. Let $f : \mathbb{D} \to \{\mathrm{Re}\, z > 0\}$ be analytic. Show that for $z \in \mathbb{D}$ we have

$$\frac{1 - |z|}{1 + |z|} |f(0)| \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|} |f(0)| \qquad \text{and} \qquad |f'(0)| \leq 2 |\mathrm{Re}\, f(0)|.$$

Zadanie 8. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and not of the form $az + b$. Show that $z, f(z), f(f(z)), \ldots, f^{\circ n}(z)$ are linearly independent over $\mathbb{C}$.

Zadanie 9. Find all injective meromorphic functions in $\mathbb{C}$.

Zadanie 10. Let U be a domain such that $\bar{\mathbb{D}} \subseteq U$ and let $f : U \to \mathbb{C}$ be analytic. Define $C_r = f(\{z : |z| = r\})$. Show that if $\mathrm{diam}(C_1) \leq 1$ then $\mathrm{diam}(C_r) \leq r$ for all $r \in [0, 1]$.

Zadanie 11. Let f be analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Show that

$$2 \int_{-1}^{1} |f(x)|^2 \mathrm{d}x \leq \int_{-\pi}^{\pi} |f(e^{it})|^2 \mathrm{d}t.$$